Stokes resolvent estimates in spaces of bounded functions

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Stokes system:

$$\begin{cases} v_t - \Delta v + \nabla q = 0 & \text{in } \Omega \times (0, T) \\ \operatorname{div} v = 0 & \text{in } \Omega \times (0, T) \\ \text{B. C.} \quad v = 0 & \text{on } \partial\Omega \\ \text{I. C.} \quad v(x, 0) = v_0 & \text{on } \{t = 0\} \end{cases}$$

in a domain $\Omega \subset \mathbf{R}^n$ with $n \geq 2$.

$v(x, t)$: unknown velocity field
$q(x, t)$: unknown pressure field
v_0 : a given initial data

$S(t) : v_0 \mapsto v(\cdot, t)(t \geq 0)$ Stokes semigroup

Analyticity results in L^∞-type spaces:

$\Omega = \mathbf{R}^n_+ \cdots$ Desch-Hieber-Prüss '01, Solonnikov '03,
Maremonti-Starita '03

$\Omega =$Admissible (e.g. bdd)$\cdots$A-Giga '11

We first consider analyticity of $S(t)$ on

$$C_{0,\sigma}(\Omega) = L^\infty\text{-closure of } \{v \in C_c^\infty(\Omega) \mid \mathrm{div}\ v = 0\}$$

A priori L^∞-estimates:

$$\sup_{0\le t\le T_0} ||N(v,q)||_\infty(t) \le C||v_0||_\infty \quad \text{for } v_0 \in C_{0,\sigma}$$

$$N(v,q)(x,t) = |v(x,t)| + t^{\frac{1}{2}}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|\partial_t v(x,t)| + t|\nabla q(x,t)|$$

Definition (Analytic semigroup)

X: Banach space, $\{T(t)\}_{t\ge 0} \subset \mathcal{L}(X)$: semigroup,
We say $T(t)$ is analytic if $\exists C > 0$ s.t.

$$\left|\left|\frac{dT(t)}{dt}\right|\right|_{\mathcal{L}(\mathrm{X})} \le \frac{C}{t} \quad \text{for } t \in (0,1].$$

Angle of analytic semigroup

If $T(t)$ has an analytic continuation to a sector $\{t \in \mathbf{C} \mid |\arg t| < \theta\}$, we say T(t) is angle θ.

Rk: $S(t)$ is angle ε on $C_{0,\sigma}$ for some positive ε (derived by an indirect method)

Problem: What is the angle of $S(t)$ on $C_{0,\sigma}$?

Resolvent problem:

$$(RS)\begin{cases} \lambda v - \Delta v + \nabla q = f & \text{in } \Omega \\ \operatorname{div} v = 0 & \text{in } \Omega \\ \text{B. C.} \quad v = 0 & \text{on } \partial\Omega \end{cases}$$

$\lambda \in \Sigma_{\vartheta,\delta}$: complex parameter

$$\Sigma_{\vartheta,\delta} = \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \vartheta,\ |\lambda| > \delta\}$$

with $\vartheta \in (\pi/2, \pi)$ and $\delta > 0$

Goal: Give a direct estimate

$$\sup_{\lambda\in\Sigma_{\vartheta,\delta}} ||M_p(v,q)||_\infty(\lambda) \le C||f||_\infty \quad \text{for } f\in C_{0,\sigma}$$

where

$$\begin{aligned} M_p(v,q)(x,\lambda) &= |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| \\ &+ |\lambda|^{n/2p}||\nabla^2 v||_{L^p(\Omega_{x,|\lambda|^{-1/2}})} \\ &+ |\lambda|^{n/2p}||\nabla q||_{L^p(\Omega_{x,|\lambda|^{-1/2}})} \end{aligned}$$

$\Omega_{x,|\lambda|^{-1/2}} = B(x,|\lambda|^{-1/2})\cap\Omega$ with $p>n$

General elliptic operators $\cdots$ Masuda '72, Stewart '74

A key idea $\cdots$ Pressure estimate:

$$\sup_{x \in \Omega} d_\Omega(x)|\nabla q(x)| \leq C_\Omega ||\nabla v||_{L^\infty(\partial\Omega)}$$

provided that Ω is strictly admissible (e.g. bdd)

q solves $\Delta q = 0$ and $\frac{\partial q}{\partial n_\Omega} = \Delta v \cdot n_\Omega$ on $\partial\Omega$.

$$\Delta v \cdot n_\Omega = \mathrm{div}_{\partial\Omega} W(v) \quad \text{on } \partial\Omega$$

by using div $v = 0$ in Ω. $-W(v) = \mathrm{curl}\ v \times n_\Omega$ $(n = 3)$

RK: We do not invoke Dirichlet B.C.

If a priori estimate

$$\sup_{x \in \Omega} d_\Omega(x) |\nabla P(x)| \leq C_\Omega ||W||_{L^\infty(\partial\Omega)}$$

holds for all solutions of the problem

$$\Delta P = 0 \quad \text{in } \Omega, \ \partial P / \partial n_\Omega = \mathrm{div}_{\partial\Omega} W \ \text{ on } \partial\Omega,$$

then we say Ω is strictly admissible.

RK

- Strictly admissible $\cdots$ C^3-bounded/exterior domains
- Not strictly admissible$\cdots$ Layer domains

Main result

Theorem 1

Let Ω be a strictly admissible, uniformly C^2-domain. For $\vartheta \in (\pi/2, \pi)$ there exist constants δ and C such that

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} ||M_p(v,q)||_{L^\infty(\Omega)}(\lambda) \leq C||f||_{L^\infty(\Omega)} \quad \text{for } f \in C_{0,\sigma}(\Omega)$$

holds with $p > n$.

Existence of sol. for $f \in C^\infty_{c,\sigma} \cdots \tilde{L}^p$-theory

Injectivity of $R(\lambda)$: If $v = R(\lambda)f = 0$, then

$$\sup_{x\in\Omega} d_\Omega(x)|\nabla q(x)| \le ||W(v)||_{L^\infty(\partial\Omega)} = 0$$

$\Rightarrow \mathrm{Ker} R(\lambda) = \{0\}$

$R(\lambda) : f \mapsto v_\lambda$ is invertible, then

$$\exists A \text{ s.t. } R(\lambda) = (\lambda - A)^{-1}$$

We call A **the Stokes operator** in $C_{0,\sigma}$

Generation results for $C_{0,\sigma}$

Theorem 2

Let Ω be a strictly admissible, uniformly C^2-domain in $\mathbf{R}^n, n \geq 2$. Then the Stokes operator A generates a C_0-analytic semigroup on $C_{0,\sigma}(\Omega)$ of angle $\pi/2$

Next consider the space

$$L^\infty_\sigma(\Omega) = \left\{ f \in L^\infty(\Omega) \mid \int_\Omega f \cdot \nabla\varphi = 0, \quad \text{for } \nabla\varphi \in L^1 \right\}$$

<u>RK</u> $C_{0,\sigma} \subset BUC_\sigma \subset L^\infty_\sigma$

<u>Approximation</u>

$\exists C_\Omega > 0$ s.t. $\forall f \in L^\infty_\sigma(\Omega)$, $\exists \{f_m\} \subset C^\infty_{c,\sigma}(\Omega)$ s.t.

$$f_m \to f \quad \text{a.e. in } \Omega$$
$$||f_m||_\infty \le C_\Omega ||f||_\infty$$

Results for L^∞_σ

Theorem 3

Let Ω be a bounded or an exterior domain with C^3 boundary. Then A generates a (non C_0)-analytic semigroup on $L^\infty_\sigma(\Omega)$ of angle $\pi/2$

RK $S(t) = e^{tA}$ is a C_0-semigroup on

$$BUC_\sigma(\Omega) = \{f \in BUC(\Omega) \mid \mathrm{div}\ f = 0, f = 0 \text{ on } \partial\Omega\}$$

Ω: bdd $\Rightarrow C_{0,\sigma} = BUC_\sigma \cdots$ Maremonti '09

RK (i) $S(t)$ is angle $\pi/2$ on $C_{0,\sigma}$ which does not follow from a priori L^∞-estimates

$$\sup_{0\le t\le T_0} ||N(v,q)||_\infty(t) \le C||v_0||_\infty \quad \text{for } v_0 \in C_{0,\sigma}$$

where

$$N(v,q)(x,t) = |v(x,t)| + t^{\frac{1}{2}}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|\partial_t v(x,t)| + t|\nabla q(x,t)|$$

RK (ii) Robin B.C., $B(v) = 0, v \cdot n_\Omega = 0$ on $\partial\Omega$

$$B(v) = \alpha v_{\mathrm{tan}} + (D(v) n_\Omega)_{\mathrm{tan}} \quad \text{with } \alpha \geq 0$$

$D(v) = (\nabla v + \nabla^T v)/2$: deformation tensor

$\Omega = \mathbf{R}^n_+ \cdots$ Saal '07

RK (iii) Ω: bdd

Energy inequality $\Rightarrow$ $S(t)$ is a bounded analytic semigroup on L^∞_σ, i.e.

$$||S(t)||_{\mathcal{L}} \quad \text{and } t||dS(t)/dt||_{\mathcal{L}}$$

are bounded in $(0, \infty)$

Ω: exterior

$S(t)$ is a bounded semigroup on L^∞_σ $\cdots$ Maremonti '12

Goal:

$$\sup_{\lambda \in \Sigma_{\vartheta,\delta}} ||M_p(v,q)||_\infty(\lambda) \leq C||f||_\infty \quad \text{for } f \in C_{0,\sigma}(\Omega)$$

Idea $\cdots$ M-S method + Pressure estimate

$$\sup_{x \in \Omega} d_\Omega(x)|\nabla q(x)| \leq C_\Omega ||\nabla v||_{L^\infty(\partial\Omega)}$$

Sketch of proof $\cdots$

(1) Localization

(2) Error estimates (key step!)

(3) Interpolation

Step1 (Localization)
Take $x_0 \in \Omega, r > 0$ and parameters $\eta \geq 1$.
Set

$$u = v\theta_0 \text{ and } p = (q - q_c)\theta_0$$

$q_c \in \mathbf{C}$: constant
θ_0: cut-off function s.t. $\theta_0 \equiv 1$ in $B_{x_0}(r)$ and $\theta_0 \equiv 0$ in $B_{x_0}((\eta + 1)r)^c$
Then (u, p) solves

$$\begin{cases} \lambda u - \Delta u + \nabla p = h & \text{in } \Omega' \\ \mathrm{div}\, u = g & \text{in } \Omega' \\ v = 0 & \text{on } \partial\Omega' \end{cases}$$

where $\Omega' = B_{x_0}((\eta + 1)r) \cap \Omega$.

L^p-estimates:

$$|\lambda|||u||_{L^p(\Omega')} + |\lambda|^{1/2}||\nabla u||_{L^p(\Omega')} + ||\nabla^2 u||_{L^p(\Omega')} + ||\nabla p||_{L^p}$$
$$\leq C_p \left(||h||_{L^p(\Omega')} + ||\nabla g||_{L^p(\Omega')} + |\lambda|||g||_{W_0^{-1,p}(\Omega')} \right)$$

where h and g contains error terms, i.e.

$$h = f\theta_0 - 2\nabla v \nabla\theta_0 - v\Delta\theta_0 + (q - q_c)\nabla\theta_0,$$
$$g = v \cdot \nabla\theta_0.$$

Step 2 (Error estimates)
How to estimate $(q - q_c)\nabla\theta_0$?
We show

$$||h||_{L^p(\Omega')} \leq Cr^{n/p}\Bigg((\eta+1)^{n/p}||f||_{L^\infty(\Omega)} + (\eta+1)^{-(1-n/p)}\Big(r^{-2}||v||_{L^\infty(\Omega)} + r^{-1}||\nabla v||_{L^\infty(\Omega)}\Big)\Bigg)$$

Note that $(\eta+1)^{-(1-n/p)} \longrightarrow 0$ as $\eta \longrightarrow \infty$ with $p > n$.

Estimates for $\theta_0 = \theta((x - x_0)/(\eta+1)r)$

$$||\theta_0||_\infty + (\eta+1)r||\nabla\theta_0||_\infty + (\eta+1)^2r^2||\nabla^2\theta_0||_\infty \leq K$$

Poincare-Sobolev-type inequality:

$$||\varphi - (\varphi)||_{L^p(\Omega_{x_0,s})} \leq Cs^{n/p}||\nabla\varphi||_{L^\infty_d(\Omega)}$$

for $p \in [1, \infty)$ where $(\varphi) = \fint_{\Omega_{x_0,s}} \varphi dx$ and

$$||f||_{L^\infty_d(\Omega)} = \sup_{x\in\Omega} d_\Omega(x)|f(x)|$$

Ex. $\Omega = B_0(1)$
$\varphi(x) = \log(1 - |x|) \in L^p$ even if $|\nabla\varphi| = d_\Omega^{-1} \notin L^p$

Ω: Strictly admissible $\Longrightarrow$ pressure estimate, i.e.

$$||\nabla q||_{L^\infty_d(\Omega)} \leq C||\nabla v||_{L^\infty(\Omega)}$$

Taking $q_c = \fint_{\Omega'} q(x)dx$ implies

$$\begin{aligned} ||q - q_c||_{L^p(\Omega')} &\leq C(\eta + 1)^{n/p} r^{n/p} ||\nabla q||_{L^\infty_d(\Omega)} \\ &\leq C(\eta + 1)^{n/p} r^{n/p} ||\nabla v||_{L^\infty(\Omega)} \end{aligned}$$

Step 3 (Interpolation)
Apply the Interpolation inequality

$$||\varphi||_{L^\infty(\Omega_{x_0,r})} \le C_I r^{-n/p} \left(||\varphi||_{L^p(\Omega_{x_0,r})} + r||\nabla\varphi||_{L^p(\Omega_{x_0,r})} \right)$$

with $\varphi = u$ and ∇u. By taking $r = |\lambda|^{-1/2}$ we have

$$M_p(v,q)(x_0,\lambda) \le C \Bigg((\eta+1)^{n/p}||f||_{L^\infty(\Omega)} + (\eta+1)^{-(1-n/p)}||M_p(v,q)||_{L^\infty(\Omega)}(\lambda) \Bigg)$$

RK: Robin B.C., $B(v) = 0, v \cdot n_\Omega = 0$ on $\partial\Omega$

$$B(v) = \alpha v_{\tan} + (D(v)n_\Omega)_{\tan} \quad \text{with } \alpha \geq 0$$

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B.C. for the localized equation:

$$B(u) = k, u \cdot n_{\Omega'} = 0 \quad \text{on } \partial\Omega'$$

with

$$k = v_{\tan} \partial\theta_0 / \partial n_{\Omega'}$$

L^p-estimates with inhomogeneous B.C.:

$$|\lambda|||u||_{L^p(\Omega')} + |\lambda|^{1/2}||\nabla u||_{L^p(\Omega')} + ||\nabla^2 u||_{L^p(\Omega')} + ||\nabla p||_{L^p}$$
$$\leq C_p\Big(||h||_{L^p(\Omega')} + ||\nabla g||_{L^p(\Omega')} + |\lambda|||g||_{W_0^{-1,p}(\Omega')}$$
$$+ |\lambda|^{1/2}||k||_{L^p(\Omega')} + ||\nabla k||_{L^p(\Omega')}\Big)$$

Summary

Blow-up arguments:

$$\sup_{0\le t\le T_0} ||N(v,q)||_\infty(t) \le C||v_0||_\infty \quad \text{for } v_0 \in C_{0,\sigma}$$

$$N(v,q)(x,t) = |v(x,t)| + t^{\frac{1}{2}}|\nabla v(x,t)| + t|\nabla^2 v(x,t)| + t|\partial_t v(x,t)| + t|\nabla q(x,t)|$$

Direct approach:

$$\sup_{\lambda\in\Sigma_{\vartheta,\delta}} ||M_p(v,q)||_\infty(\lambda) \le C||f||_\infty \quad \text{for } f \in C_{0,\sigma}$$

$$M_p(v,q)(x,\lambda) = |\lambda||v(x)| + |\lambda|^{1/2}|\nabla v(x)| + |\lambda|^{n/2p}||\nabla^2 v||_{L^p(\Omega_{x,|\lambda|^{-1/2}})} + |\lambda|^{n/2p}||\nabla q||_{L^p(\Omega_{x,|\lambda|^{-1/2}})}$$